Modeling and Analysis of Time-Dependent Stress Accelerated Life Data

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SUMMARY & CONCLUSIONS

The Cumulative Damage model is examined in this paper in two ways. First for analyzing data obtained from accelerated tests where the applied stress is time-dependent (for example step-stress) and second for making reliability predictions when the service stress is time-dependent (even when the test stress is not time-dependent). For example, the reliability of an aircraft engine, which operates under different stresses during take-off, cruising and landing, can be determined. The cumulative damage model is formulated by assuming the Weibull as the underlying life distribution, and the Arrhenius and inverse power law relationships are considered. The parameters of the model are estimated using maximum likelihood. An algorithm has been developed for the solution of this model, and implemented in a recently released software package, ALTA 6 PRO, specific to accelerated life data analysis.

The solution to this model provides the engineers with an opportunity to expand their selection of test conditions when testing products, and to better model actual usage conditions.

1. INTRODUCTION

The life of most products depends on the operating stresses they experience in service. Therefore, the need to relate life and stress becomes a necessity in order to perform reliability predictions. In addition, the stresses experienced by a product in service are usually time-dependent. Accelerated life tests provide the data from which a life-stress relationship of a product can be obtained. Such tests are becoming increasingly popular in today’s industry due to the need to quickly obtain life data. However, the rate of increase of this popularity has been very slow due to the limited options available at this time for performing adequate life data analyses on data obtained from accelerated tests. Most of the tools available today for accelerated life data analyses are limited to performing analyses on data obtained from constant stress tests, and extrapolating life for constant use stress conditions. In most practical applications however, there is a need to perform accelerated tests under time-varying stresses in order to further reduce test time, as well as estimate life under constantly changing operating conditions. The theory for such cases has been developed for some time now, the means of its application however, has been a subject of wishful thinking.

The value of accelerated life tests extends beyond the need to reduce testing time. One piece of information they can provide, which is commonly overlooked, is the fact that in an accelerated test, life is associated with stress. In other words, an accelerated test can also be described as a life-stress test, which can provide very valuable information regarding the performance of the product in the field under different and changing operating conditions. Once a stress profile has been determined for these operating conditions, and by utilizing the data obtained from an accelerated test, the reliability of the product can be determined using the cumulative damage model.

Traditionally, accelerated tests using a time-varying stress application have been used to assure failures quickly. This is highly desirable given the pressure on industry today to shorten new product introduction time. The most basic type of time-varying stress test is a step-stress test. In step-stress accelerated testing, the test units are subjected to successively higher stress levels in predetermined stages, and thus a time-varying stress profile. The units usually start at a lower stress level and at a predetermined time, or failure number, the stress is increased and the test continues. The test is terminated when all units have failed, or when a certain number of failures are observed, or when a certain time has elapsed. Step-stress testing can substantially shorten the reliability test's duration. In addition to step-stress testing, there are many other types of time-dependent stress profiles that can be used in accelerated life testing.

When dealing with data from accelerated tests with time-varying stresses, the life-stress model must take into account the cumulative effect of the applied stresses. Such a model is commonly referred to as a cumulative damage or cumulative exposure model. Nelson (Ref. 1) defines and presents the derivation and assumptions of such a model.

This paper describes the statistical models, methodology and a computer program used to analyze such data and perform inferences on the life of a product based on its operating stress profile (constant or time-dependent). Utilizing the maximum likelihood estimation method, the likelihood function was formulated and solved for complete and suspended data and for the Weibull as the underlying life distribution. The Arrhenius and inverse power law models were assumed for the life-stress relationship. Confidence bounds on the estimates were quantified using the variance-covariance matrix.

2. NOTATION

β	Weibull shape parameter
η	Weibull scale parameter
T_F	exact time-to-failure
T_S	right censored time-to-failure
x(t)	stress as a function of time
B	parameter of the Arrhenius relationship associated with the activation energy
C	parameter of the Arrhenius relationship
a	parameter of the IPL relationship
n	parameter of the IPL relationship
$\Lambda()$	log-likelihood function

3. ASSUMPTIONS

The following two assumptions were made prior to formulating the model:

1. Common shape parameter. This assumption is valid when similar modes of failure are accelerated.
2. Stress can be any integrable function of time.

4. BACKGROUND THEORY

Accelerated life models consist of an underlying failure distribution and a life-stress relationship. A percentile of the failure distribution is chosen to be represented by the stress-life relationship. The objective then becomes to obtain the parameters of the failure distribution and the stress-life relationship. In this paper, the Weibull distribution will be considered and the maximum likelihood parameter estimation method will be utilized. A general likelihood function is to be formulated for these two distributions. The likelihood function is given by

$$L = \prod_{i=1}^{F} f(T_{F,i}) \prod_{j=1}^{S} R(T_{S,j}). \tag{1}$$

Taking the natural logarithm of eq. (1) simplifies the optimization. The log-likelihood function is given by

$$\ln(L) = \Lambda = \sum_{i=1}^{F} \ln\left[f(T_{F,i})\right] + \sum_{j=1}^{S} \ln\left[R(T_{S,j})\right] \tag{2}$$

The model will be formulated in such a way where eq. (2) will be a function of stress by expressing the *pdf* and reliability functions in terms of the stress. In addition, the stress will be time-dependent.

4.1 *Cumulative Damage*

Nelson (Ref. 1) describes the cumulative damage model. A small overview of the model for any time-dependent stress is given in this section. The model will be formulated using the Weibull distribution. For the Weibull distribution it is assumed that the scale parameter, η, is a function of stress. In the case of a time-dependent stress, the scale parameter is also a function of time. In this case, the cumulative damage-Weibull reliability function is given by (Ref. 1):

$$R(t, x(t)) = e^{-\left[\int_0^t \frac{1}{\eta(u, x(u))} du\right]^{\beta}}. \tag{3}$$

Any life-stress relationship can be used to express η as a function of stress. The Arrhenius and inverse power law relationships are considered next.

4.1.1 *Cumulative Damage Arrhenius-Weibull Model*

The Arrhenius relationship is given by,

$$\eta(x) = C \cdot e^{\frac{B}{x}}.$$

Eq. (3) can be written as

$$R(t, x(t)) = e^{-\left[\int_0^t \frac{1}{C} \cdot e^{-\frac{B}{x(u)}} du\right]^{\beta}}. \tag{4}$$

Eq. (4) is the cumulative damage Arrhenius-Weibull reliability function, and it can be used to estimate the reliability of a product at a given time under a time-dependent stress, and given the parameters β, *C* and *B*. Note that eq. (4) can be used to estimate the reliability of a product at a given time under a constant stress as well. When the stress is constant, performing the integration in eq. (4), yields the Arrhenius-Weibull reliability function,

$$R(t, x) = e^{-\left(\frac{t}{C} \cdot e^{-\frac{B}{x(u)}}\right)^{\beta}}.$$

4.1.2 *Cumulative Damage IPL-Weibull Model*

The IPL relationship is given by,

$$\eta(x) = \left(\frac{a}{x}\right)^{n}.$$

Eq. (3) can be written as

$$R(t, x(t)) = e^{-\left[\int_0^t \left(\frac{x(u)}{a}\right)^{n} du\right]^{\beta}}. \tag{5}$$

Eq. (5) is the cumulative damage IPL-Weibull reliability function, and it can be used to estimate the reliability of a product at a given time under a time-dependent stress, and given the parameters β, *a* and *n*.

4.1.3 *Parameter Estimation*

The parameters of the cumulative damage Arrhenius-Weibull and IPL-Weibull models can be estimated using the maximum likelihood estimation method, from data obtained from either a constant stress test, or a time-dependent stress test. A general likelihood function can be formulated, which will include both cases of time-dependent and constant stress data.

The likelihood function for the cumulative damage Arrhenius-Weibull model is given by:

$$\Lambda=\sum_{i=1}^{F}\ln\left\{\beta\cdot\frac{1}{C}\cdot e^{-\frac{B}{x(T_{F,i})}}\cdot\left[\int_0^{T_{F,i}}\frac{1}{C}\cdot e^{-\frac{B}{x(u)}}du\right]^{\beta-1}\right\}$$
$$-\sum_{i=1}^{F}\left\{\left[\int_0^{T_{F,i}}\frac{1}{C}\cdot e^{-\frac{B}{x(u)}}du\right]^{\beta}\right\} \quad (6)$$
$$-\sum_{j=1}^{S}\left\{\left[\int_0^{T_{S,j}}\frac{1}{C}\cdot e^{-\frac{B}{x(u)}}du\right]^{\beta}\right\}.$$

The likelihood function for the cumulative damage IPL-Weibull model is given by:

$$\Lambda=\sum_{i=1}^{F}\ln\left\{\beta\cdot\left[\frac{x(T_{F,i})}{a}\right]^{n}\cdot\left[\int_0^{T_{F,i}}\left(\frac{x(u)}{a}\right)^{n}du\right]^{\beta-1}\right\}$$
$$-\sum_{i=1}^{F}\left\{\left[\int_0^{T_{F,i}}\left(\frac{x(u)}{a}\right)^{n}du\right]^{\beta}\right\} \quad (7)$$
$$-\sum_{j=1}^{S}\left\{\left[\int_0^{T_{S,j}}\left(\frac{x(u)}{a}\right)^{n}du\right]^{\beta}\right\}.$$

Note in eqs (6) and (7) that any stress profile, *x(t)*, can be used. Nelson (Ref. 1) identifies the following types of time-dependent stress loading as some of the most commonly used in practical applications.

1. Step stress.
2. Ramp stress.
3. Cyclical stress.
4. Randomly varying stress over time (stochastic loading).
5. Non-repeating pattern.

The developed algorithm used in this paper is independent of the form of the stress profile. In other words, engineers can use different stress profiles and perform the analysis for a wide range of stress profiles, including the above-mentioned stress loading, as well as for constant stress loading scenarios.

4.2 *Confidence Bounds*

The confidence bounds on the parameters and a number of other quantities such as the reliability and the percentile can be obtained based on the asymptotic theory for maximum likelihood estimates, most commonly referred to as the Fisher matrix bounds. The variance/covariance matrix is obtained by taking the inverse of the Fisher matrix.

$$\begin{bmatrix} Var(\beta) & Cov(\beta,a_1) & . & . & . & Cov(\beta,a_m) \\ Cov(\beta,a_1) & Var(a_1) & & & & . \\ . & & & . & & . \\ . & & & & . & . \\ Cov(\beta,a_m) & . & & . & . & Var(a_1) \end{bmatrix}=$$
$$=\begin{bmatrix} \frac{\partial^2\Lambda}{\partial\beta^2} & \frac{\partial^2\Lambda}{\partial\beta\partial a_1} & . & . & . & \frac{\partial^2\Lambda}{\partial\beta\partial a_m} \\ \frac{\partial^2\Lambda}{\partial\beta\partial a_1} & \frac{\partial^2\Lambda}{\partial a_1^2} & & & & . \\ . & & & . & & . \\ . & & & & . & . \\ \frac{\partial^2\Lambda}{\partial\beta\partial a_m} & & . & . & . & \frac{\partial^2\Lambda}{\partial a_m^2} \end{bmatrix}^{-1}$$

Nelson (Ref. 1) and Lawless (Ref. 2) provide an extensive coverage on the Fisher matrix bounds.

5. THE ALGORITHM

Several algorithms have been used in the past for the maximization of the log-likelihood function such as the Newton search (Refs.1, 2), genetic algorithms, annealing method, etc. The problem was approached with a form of the Newton search method, which is closely related to the Quasi-Newton method (Ref. 3). This method was chosen because it is versatile, reliable, and provides quick convergence. The method maximizes the log-likelihood function (eq. (6) or eq. (7)) by taking Newton steps in order to bring its partial derivatives to zero. The full Newton step is always performed, since a quadratic convergence can be achieved once near the solution. At each iteration a check is performed to determine if the proposed step reduces the log-likelihood function. If not, a backtrack along the Newton direction is performed until an acceptable step is achieved. As in every optimization algorithm, the initial guesses for the parameters are very crucial. For this reason, much of the research focused in obtaining them. The initial guesses are obtained from the supplied data, thus increasing the probability of convergence to a global minimum (if it exists), and decreasing the number of iterations.

The second part of the algorithm is the integration. The algorithm was developed with the flexibility of solving for any supplied (and integrable) user-defined stress profile. For this, the Guassian Quadratures integration method was utilized. In addition, the algorithm has the ability to identify certain functions, and perform the integration analytically, for better accuracy.

6. EXAMPLES

6.1 *Constant Stress Test with Time-Dependent Use Stress*

Consider the data summarized in Table 1. These data illustrate a typical 3-stress level, constant stress accelerated test. The stress in this example is temperature, and the Arrhenius relationship will be used.

Table 1: Failure data

	Temperature		
	406 K	416 K	426 K
Time-to-Failure	248	164	92
	456	176	105
	528	289	155
	731	319	184
	813	340	219
		543	235

In this case, the stress profile is constant, and each stress level can be considered a different profile. Specifically,

$$x_1(t) = x_1 = 406,$$
$$x_2(t) = x_2 = 416, \qquad (8)$$
$$x_3(t) = x_3 = 416.$$

Using eq. (6), the data of Table 1, and the stress profile given by eq. (8), the parameters of the model, which maximize this log-likelihood function, can be obtained. The estimated parameters obtained using the developed algorithm are:

β = 2.9658; B = 1.0680E+4; C = 2.3966E-9.

Several types of information about the model as well as the data can be obtained from a probability plot. For example, the choice of an underlying distribution and the assumption of a common slope (shape parameter) can be examined. In this example, the linearity of the data supports the use of the Weibull distribution. In addition, the data appear parallel on this plot, therefore reinforcing the assumption of a common beta.

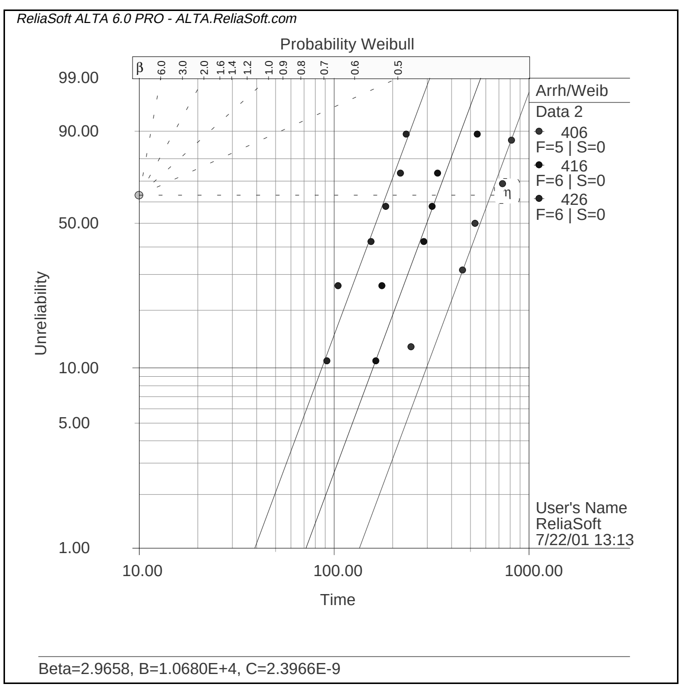

Figure 1: Weibull probability plot of the three test stresses.

Further statistical analysis can and should be performed for these purposes as well. An extensive coverage of assessing the model assumptions is provided by Nelson (Ref. 1). For example, a likelihood ratio test can be performed to validate the assumption of a common shape parameter across the three stress levels.

This particular component is a part of a production line, which operates continuously except when the line is shut down during shift changes. Therefore, the use stress for this product is time-dependent (Figure 2) and is given by:

$$x(t) = \begin{cases} (298 + 116 \cdot t)K, & 0 \le t < 0.5\ hr, \\ 356K, & 0.5 \le t < 8\ hr, \\ (356 - 116 \cdot (t-8))K, & 8 \le t < 8.5\ hr. \end{cases} \qquad (9)$$

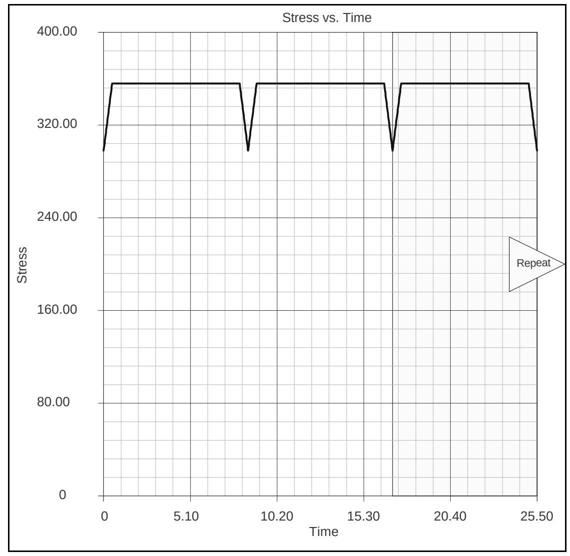

Figure 2: Time-dependent use stress profile.

The reliability of this product under the use stress conditions given by eq. (9) can be estimated using eq. (4). Figure 4 is the reliability plot of this product under its actual operating stress given by eq. (9).

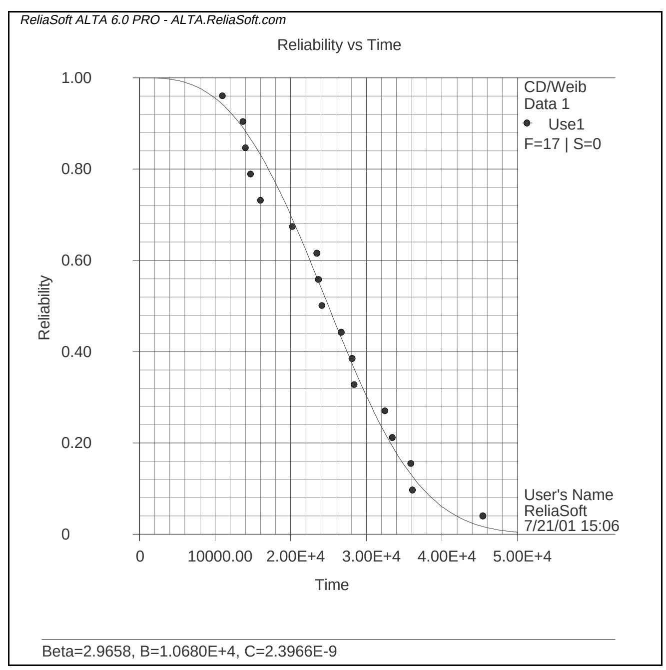

Figure 3: Reliability vs. Time under a time-dependent use stress.

Traditionally, the reliability would have been estimated using a constant stress usage level. For example, in this case it could be assumed that the operating stress is 356K. The reliability for a use stress of 356K is plotted in Figure 4 next to the reliability under the actual use conditions. The difference between the two reliability curves in Figure 4 is indicative of the error that can be caused by assuming a constant use stress when in fact the use stress is time-dependent.

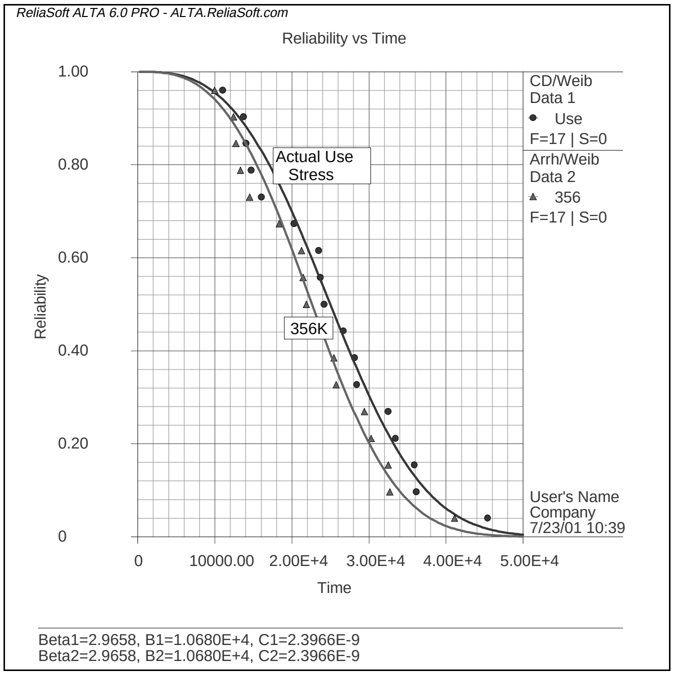

Figure 4: Comparison of the reliability between constant and actual time-dependent operating conditions.

6.2 *Step-Stress Test*

As a second example we will consider a typical step-stress test. In this test multiple stresses were applied simultaneously to a particular component in order to precipitate failures more quickly than they would occur under normal use conditions. The combination of the applied stresses was quantified in terms of a "percentage stress" as compared to typical stress levels (or assumed field conditions). In this scenario, the typical stress (field or use stress) was defined as 100% and any combination of the test stresses was quantified as a percentage over the typical stress. For example, if the combination of stresses on test was determined to be two times higher than typical conditions, then the stress on test was said to be at 200%.

The stress profile of the test is shown in Figure 5, and is given by,

$$x(t) = \begin{cases} 125, & 0 \le t < 200, \\ 175, & 200 \le t < 300, \\ 200, & 300 \le t < 350, \\ 250, & 350 \le t < 375. \end{cases} \quad (10)$$

The data obtained from the test are given in Table 2.

Table 2: Test data

State	Time-to-Failure
F	252
F	280
F	320
S	328
F	335
F	354
F	361
F	362
F	368
S	375
S	375
S	375

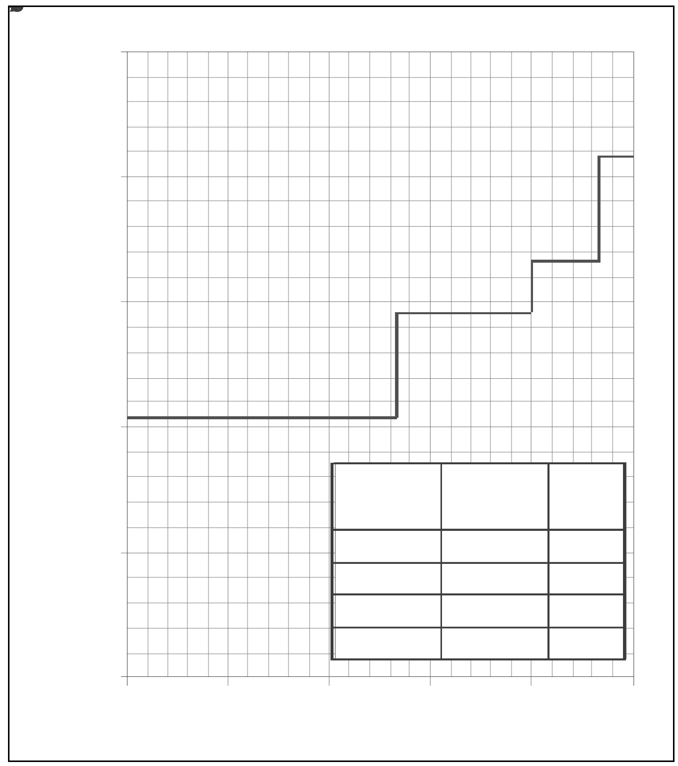

Figure 5: Step-stress profile.

Using the data in Table 2, the stress profile given by eq. (10), and assuming the IPL-Weibull model, the parameters that maximize eq. (7) are:

β = 5.7513; a = 1.0779E+4; n = 1.3208.

Once the parameters are estimated, further analysis on the data can be performed. First, a Weibull probability plot can be obtained at the use level of 100%, shown in Figure 6. In this plot it can be seen that the Weibull distribution provides a good fit for the data.

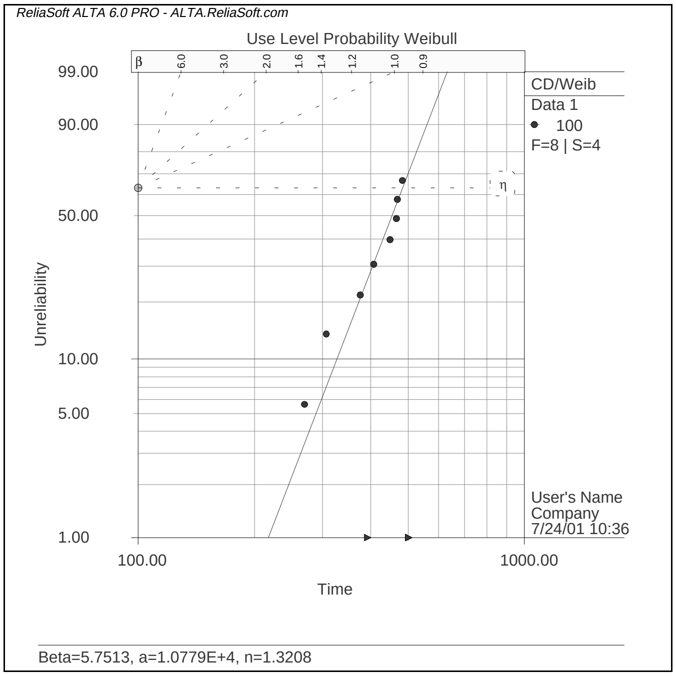

Figure 6: Weibull propability plot at a use level of 100%.

The Life vs. Stress plot is a very common plot for the analysis of accelerated data, as shown in Figure 7. In such a plot, life is plotted for a range of stress levels in a log-log scale (when using the IPL relationship). Each line in Figure 7 is a constant reliability line.

Life vs. Stress plots can be very useful in assessing the effect of each stress to a product's failure. The 90% reliability is plotted in Figure 7. This provides the flexibility of obtaining the 90% life for any stress level with the corresponding confidence bounds.

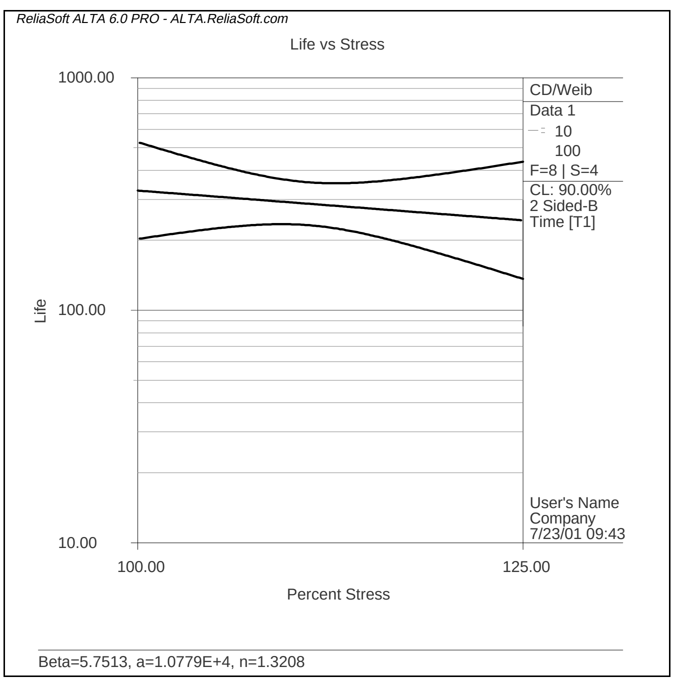

Figure 7: Life vs. Stress plot

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