Modeling and Analysis of Repairable Systems with General Repair

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SUMMARY & CONCLUSIONS

The commonly used models for analyzing repairable systems data are *perfect renewal processes* (PRP), corresponding to perfect repairs, and *nonhomogeneous Poisson processes* (NHPP), corresponding to minimal repairs. However, most repair activities may realistically not result in such two extreme situations but in a complicated intermediate one (general repair or imperfect repair/maintenance). In this paper, we explore the *general renewal processes* (GRP) to model and analyze complex repairable systems with various degrees of repair. A general likelihood function formulation for single and multiple repairable systems is presented for estimation of the GRP parameters. Confidence bounds based on the Fisher information matrix are also developed. The practical use of the proposed statistical inference is demonstrated by two examples, and the results show that our proposed method is a very promising and efficient approach with the potential of becoming very useful in industry and of leading to further generalization of repairable systems analysis.

1 INTRODUCTION

Repairable systems receive repair/maintenance actions that restore system components when they fail. These actions change the overall makeup of the system and affect the system behavior differently due to different maintenance approaches. Basically, there are two major categories: corrective maintenance or preventive maintenance. Each can be classified in the terms of the degree to which the operating condition of an item is restored by maintenance in the following way [1][2]:

a) Perfect repair or maintenance: a maintenance action that restores the system operating condition to be "as good as new."
b) Minimal repair or maintenance: a maintenance action that restores the system operating state to be "as bad as old."
c) Imperfect repair or maintenance: a maintenance action that restores the system operating state to be somewhere between as good as new and as bad as old.
d) Worse repair or maintenance: a maintenance action that makes the operating condition worse than that just prior to failure.
e) Worst repair or maintenance: a maintenance action that makes the system fail or break down undeliberately.

Earlier studies and results in this field usually assumed that the system after corrective or preventive maintenance is as good as new (perfect maintenance) or as bad as old (minimal maintenance). These two assumptions are often found very limited uses in practical applications. Many maintenance activities may realistically not result in either of these two extreme situations but in a complicated intermediate one. That is, when the system is maintained correctively or preventively, its failure rate is somewhere between as good as new and as bad as old. Imperfect maintenance is the concept that maintenance actions do not bring the system to an as good as new condition but rather bring the state of a failed system to a level that is somewhere between new and prior to failure.

Recently, modeling and analysis of repairable systems with general repair have drawn a lot of attention in reliability and maintenance work. However, as Guo, Ascher and Love[3] noticed, too much attention is paid to the invention of new models, with little thought, it seems, as to their applicability. Too little attention is paid to data collection and considering the usefulness of models for solving problems through model fitting and validation. The scarcity of these works can be explained by the complexity of the likelihood function. Kijima[4][5] suggested two possible probabilistic models to address a very general assumption regarding the system repair condition called *general renewal process*. Kijima model I assumes that repairs only fix the wearout and damage created in the last period of operation; Kijima model II assumes that repairs fix all of the wearout and damage accumulated up to the current time. Because of the mathematical complexity of the g-renewal equation, the closed form solution of the equation is not available, and even numerical solutions are extremely difficult to obtain. Based on Kijima and Sumita's work, Kaminskiy and Krivtsov[6] proposed an approximate solution to Kijima model I using the Monte Carlo (MC) simulation technique. Yanez et. al [7] combined MC simulation with numerical method to solve maximum likelihood (ML) estimation for Kijima model I. The realism of Kijima model I's assumption is often questioned. In practice, the *n*th repair not only can remove the damage incurred during the time between the (*n-1*)th and *n*th failures, but also can fix the cumulative damage incurred during the time from the *1*th failure to (*n-1*)th failure. On the other hand, the Monte Carlo approach was developed mainly for the cases where a large set of data is available, and the accuracy for estimation of the time to first failure (TTFF) distribution depends on the availability of such data. It would be very difficult to obtain the large set of data in many industries such as nuclear, chemical and petrochemical. Besides the need for large amounts of data, the approach is extremely time-consuming and slow in order to estimate the parameters [7].

In this paper, Kijima model II is introduced to model complex repairable systems, a general likelihood function formulation for single and multiple systems with the time truncated data and failure truncated data is presented for the estimation of the parameters. The remainder of this paper is organized as follows: In section 2, we provide an overview of the probabilistic models for repairable systems. In section 3, we propose a new approach and ML solution of the proposed approach and revisit some examples in [8] using our proposed method. Our conclusions and directions for future research are in section 4.

1.1 Assumptions

- TTFF distribution is known and can be estimated from the available data
- The repair time is assumed to be negligible so that the failures can be viewed as point processes.

2 BASIC ANALYSIS APPROACHES FOR REPAIRABLE SYSTEMS

2.1 Renewal Process and Homogeneous Poisson Process

If a system in service can be repaired to an as good as new condition following each failure, then the failure process is called a renewal process. For renewal processes, the times between failures are independent and identically distributed.

A special case of this is the homogeneous Poisson process (HPP), which has independent and exponential times between failures. A counting process is a homogenous Poisson process with parameter $\lambda>0$ if :

- $N(0)=0$
- the process has independent increments
- the number of failures in any interval of length t is distributed as a Poisson distribution with parameter λt

There are several implications to this definition of the Poisson process. The distribution of the number of events in $(t_1, t_2]$ has the Poisson distribution with parameter $\lambda(t_2-t_1)$. Therefore, the probability mass function is:

$$P[N(t_2)-N(t_1)=n]=\frac{[\lambda(t_2-t_1)]^x e^{-\lambda(t_2-t_1)}}{n!} \quad n=0,1,2, \quad (1)$$

The expected number of failures by time t is $\Lambda(t)=E[N(t)]=\lambda t$, where λ is often called the failure intensity or rate of occurrence of failure (ROCOF). The intensity function is therefore $u(t)=\Lambda'(t)=\lambda$. If X_1, X_2, …, are independent and identically distributed exponential random variables, then $N(t)$ corresponds to a Poisson process.

2.2 Nonhomogeneous Process (NHPP)

As a general class of well-developed stochastic process models in reliability engineering, nonhomogeneous Poisson process models have been successfully used in studying hardware reliability problems. NHPP models are especially useful to describe failure processes that possess certain trends, such as reliability growth or deterioration. The cumulative number of failures up to time t, $N(t)$, can be described by a NHPP. For the counting process $\{N(t), t\geq 0\}$ modeled by NHPP, $N(t)$ follows a Poisson distribution with parameter $\Lambda(t)$. The probability that $N(t)$ is a given integer n is expressed by:

$$P\{N(t)=n\}=\frac{[\Lambda(t)]^n}{n!}e^{-\Lambda(t)}, \; n=0,1,2,\ldots$$

$\Lambda(t)$ is the mean value function. The function $\Lambda(t)$ describes the expected cumulative number of failure behavior.

The underlying assumptions of the NHPP are:

- $N(0) = 0$
- $\{N(t), t \geq 0\}$ has independent increments
- $P\{N(t+h) - N(t) = 1\} = u(t) + o(h)$
- $P\{N(t+h) - N(t) \geq 2\} = o(h)$

The probability of exactly n events occurring in the interval $(a, b]$ is given by:

$$P[N(b)-N(a)=n]=\frac{\left[\int_a^b u(t)dt\right]^n e^{-\int_a^b u(t)dt}}{n!} \quad \text{for } n=0,1,\ldots. \quad (2)$$

$o(h)$ denotes a quantity that tends to zero for small h. The function $u(t)$ is the failure intensity. Given $u(t)$, the mean value function $\Lambda(t)=E[N(t)]$ satisfies:

$$\Lambda(t)=\int_0^t u(s)ds$$

Inversely, knowing $\Lambda(t)$, the failure intensity at time t can be obtained by:

$$u(t)=\frac{d\Lambda(t)}{dt}$$

One of the most common forms of the failure intensity used in reliability analysis of repairable systems, the Crow (AMSAA) model, is as follows:

$$u(t)=\lambda\beta t^{\beta-1} \quad (3)$$
$$E\left(N(t)\right)=\lambda t^{\beta}$$

where,

$N(t)$ = number of observed failures in $(0, \; t)$

$u(t)$ = failure intensity (sometimes called the "instantaneous failure rate")

λ, β = model parameters ($\lambda>0, \beta>0$)

2.2.1 Estimation for Failure Truncated Data

In general, the process is said to be failure truncated if it is observed until a fixed number of failures have occurred. It is said to be time truncated if it is observed for a fixed length of time. For an item undergoing reliability growth testing, if the failure process follows the Weibull process and testing data are truncated at the n^{th} failure with $0<t_1<t_2<\ldots<t_n$ denoting the successive failure times, the likelihood function is:

$$L(t_1,t_2,\ldots,t_n,\lambda,\beta)=\lambda^n\beta^n\exp(-\lambda t_n^{\beta})\prod_{i=1}^{n}t_i^{\beta-1}$$

Maximum likelihood estimates (MLE) for β and λ are:

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n-1} \ln(t_n/t_i)} \quad (4)$$

$$\hat{\lambda} = \frac{n}{t_n^{\hat{\beta}}} \quad (5)$$

2.2.2 Estimation for Time Truncated Data

A process is said to be time truncated if it is observed for a fixed length of time. Some of the estimation and inference procedures are quite similar to those for failure truncated data. For an item undergoing reliability growth testing, if the failure process follows a Weibull process and testing data are truncated at time T with $0<t_1<t_2<\ldots\leq T$ denoting the successive failure times, the likelihood function is:

$$f(t_1, t_2, ..., t_n, \lambda, \beta) = \lambda^n \beta^n \exp(-\lambda T^\beta) \prod_{i=1}^{n} t_i^{\beta-1}$$

Thus, the maximum likelihood estimates of β and λ are:

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n} \ln(T/t_i)} \quad (6)$$

$$\hat{\lambda} = \frac{n}{T^{\hat{\beta}}} \quad (7)$$

2.2.3 Estimation for Multiple System Data

When the data come from multiple copies of the product, it is necessary to modify the calculation of $\hat{\lambda}$ and $\hat{\beta}$. Suppose the q^{th} system is observed continuously from time S_q to time T_q (q = 1, ..., k). Then the maximum likelihood estimates of λ and β are values $\hat{\lambda}$ and $\hat{\beta}$ given by [7]:

$$\hat{\lambda} = \frac{\sum_{q=1}^{K} N_q}{\sum_{q=1}^{K} (T_q^{\hat{\beta}} - S_q^{\hat{\beta}})} \quad (8)$$

$$\hat{\beta} = \frac{\sum_{q=1}^{K} N_q}{\hat{\lambda} \sum_{q=1}^{K} (T_q^{\hat{\beta}} \ln T_q - S_q^{\hat{\beta}} \ln S_q) - \sum_{q=1}^{K} \sum_{i=1}^{N_q} \ln X_{iq}} \quad (9)$$

2.3 Generalized Renewal Process (Kijima Model I)

Kijima et al. [5,6] developed an imperfect repair model by using the idea of the virtual age process of a repairable system. If the system has the virtual age $V_{n-1} = y$ immediately after the (n-1)th repair, the nth failure time X, is assumed to have the distribution function:

$$\Pr(X_n < X \mid V_{n-1} = y) = \frac{F(X+Y) - F(y)}{1 - F(Y)}$$

Where $F(X)$ is the distribution function of the time to failure of a new system. The real age of the system is $S_n = \sum_{i=1}^{n} X_i$. Let q be the degree of the nth repair where $0 \leq q \leq 1$ and the virtual age of a new system $V_0 = 0$ They construct such a repair model: the nth repair cannot remove the damage incurred before the (n-1)th repair. It reduces the additional age X_n to qX_n. Accordingly, the virtual age after the nth repair becomes:

$$V_n = V_{n-1} + qX_n$$

Thus:

$$V_n = q(X_1 + X_2 + \cdots + X_n)$$

The expected number of failures in $[0, t]$ is given by a g-renewal equation [4]:

$$H(t) = \int_0^t \left(g(\tau \mid 0) + \int_0^\tau h(x) g(\tau - x \mid x) dx \right) d\tau \quad (10)$$

where $g(\tau \mid x) = \frac{f(t+qx)}{1-F(qx)}, \; t, x \geq 0$

Because it is impossible to obtain the closed form solution, Kaminskiy and Krivtsov [5] proposed an approximate solution to the g-renewal equation using the Monte Carlo simulation technique. Let a sample of independent and identical repairable systems be observed at discrete time intervals over the time period $[0, t]$. An empirical cumulative intensity function $\Lambda_e(t)$ can be estimated at the end of t_i th interval, $i = 1, \cdots, n$:

$$\Lambda_e(t) = \frac{1}{k} \sum_{j=1}^{k} N_j(t_i)$$

Where $N_j(t_i)$ is the number of failures of the j th system in $[0, \; t_i]$ and k is the number of systems at $t = 0$. A Monte Carlo generated cumulative intensity function is $\Lambda_{mc}(F(\alpha_1, \alpha_2, \cdots, \; \alpha_n, \tau), \; q, \; t)$. Let $[t/n]$ be the greatest integer less than or equal to t/n. Then the solution of $\min_{\alpha_1, \cdots, \alpha_n, q} \left(\sum_{i=1}^{[t/n]} (\Lambda_e(t_i) - \Lambda_{mc}(F(\alpha_1, \cdots, \alpha_n, \; \tau), \; q, \; t_i))^2 \right)$ provides the nonlinear least square estimate of $\alpha_1, \alpha_2, \cdots, \; \alpha_n$, and q.

3 KIJIMA MODEL II AND MAXIMUM LIKELIHOOD ESTIMATES

3.1 Model Description

Consider a repairable system, observed from time $t = 0$. Denote by $t_1, t_2, \cdots$ the successive failure times and let the times between failures be denoted by $x_1, x_2, \; \cdots$. Thus we have:

$$x_i = t_i - t_{i-1}, \quad i = 1, \; 2, \; \cdots$$

where for convenience we define $t_0 \equiv 0$. The sequence $t_1, \; t_2, \cdots$ of failure times and the sequence $x_1, \; x_2, \cdots$ of inter-arrival times thus contain exactly the same information about a particular realization of the process.

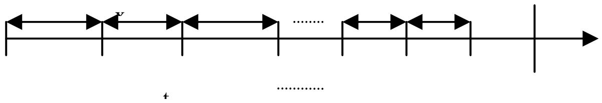

Figure 1. A Repairable System Structure

Previous research assumes that the nth repair can remove the damage incurred only during the time between the (n-1)th and nth failures. In practice, not only does the nth repair depend on (n-1)th repair, but also it depends on all previous repair. We assume that the repair action could remove all damage accumulated up to nth failure, accordingly, the virtual age after the nth repair becomes:

$$V_n = q(V_{n-1} + x_n)$$

where q is the degree of the nth repair where $0 \le q \le 1$, thus:

$$\begin{aligned} V_0 &= 0 \\ V_1 &= qx_1 \\ V_2 &= q(qx_1 + x_2) \\ &\vdots \\ V_n &= q(q^{n-1}x_1 + q^{n-2}x_2 + \cdots + x_n) \end{aligned} \tag{11}$$

If the system has the virtual age $V_{n-1} = y$ immediately after the (n-1)th repair, the nth failure time X is distributed according to the following cumulative density function (*cdf*):

$$F(X \mid V_{n-1} = y) = \frac{F(X+Y) - F(y)}{1 - F(Y)} \tag{12}$$

Obviously, $q = 0$ corresponds to a perfect repair (RP, as good as new) while $q = 1$ leads to a minimal repair (NHPP, as bad as old). The case of $0 < q < 1$ corresponds to an imperfect repair (better –than –old –but –worse –than –new) while $q > 1$ leads to worse or worst repair (worse than old). The case of $q < 0$ suggested a system restored to a condition of better than new. Physically speaking, therefore, q can be an index for repair effectiveness

3.2 Maximum Likelihood Estimation of the Parameters

The Monte Carlo approach proposed by Kaminskiy and Kivtsov provides a simulation method for statistical estimation of the GRP, and it has been used in the automotive industry [6]. However, this approach needs to estimate the distribution of the TTFF from a large amount of data and it will take a very long time to estimate the parameters. For these reasons, using MLE to estimate the GRP parameters is preferable.

3.2.1 Single Repairable System

A maximum likelihood estimation method is possible for cases in which there is reasonably enough data available. Let $t_1, t_2, \cdots, t_n$ be the inter-arrival time between failure $i-1$ and i, assuming a Weibull distribution for TTFF, the nth failure time t_i is distributed according to the following *cdf*:

$$\begin{aligned} F(t_i \mid V_{i-1} = v_{i-1}) &= \frac{F(x_i + v_{i-1}) - F(v_{i-1})}{1 - F(v_{i-1})} \\ &= \frac{e^{-\lambda v_{i-1}^{\beta}} - e^{-\lambda (x_i + v_{i-1})^{\beta}}}{e^{-\lambda v_{i-1}^{\beta}}} \\ &= 1 - e^{-\lambda[(x_i + v_{i-1})^{\beta} - v_{i-1}^{\beta}]} \end{aligned} \tag{13}$$

Thus the conditional probability distribution function (*pdf*) of t_i is:

$$\begin{aligned} f(t_i \mid t_{i-1}, t_i, \cdots, t_1) &= f(t_i \mid t_{i-1}) \\ &= \lambda\beta(x_i + v_{i-1})^{\beta-1} \exp[-\lambda[(x_i + v_{i-1})^{\beta} - v_{i-1}^{\beta}]] \end{aligned}$$

Where $t_i > t_{i-1}$

The corresponding likelihood is:

$$\begin{aligned} L\{\text{data}|\lambda, \beta, q\} &= f(t_1) f(t_2 \mid t_1) \cdots f(t_n \mid t_{n-1}) R(T \mid t_n) \\ &= \lambda^n \beta^n [e^{-\lambda[(T - t_n + v_n)^{\beta} - v_n^{\beta}]}]^{\delta} \prod_{i=1}^{n} [(x_i + v_{i-1})^{\beta-1} e^{-\lambda[(x_i + v_{i-1})^{\beta} - v_{i-1}^{\beta}]}] \end{aligned} \tag{14}$$

Where $\delta = \begin{cases} 0 & \text{If the test is failure truncated} \\ 1 & \text{If the test is time truncated} \end{cases}$

Taking the natural log on both sides:

$$\begin{aligned} \log L\{\text{data}|\lambda,\beta,q\} &= n(\ln\lambda + \ln\beta) \\ &- \lambda\delta[(T - t_n + v_n)^{\beta} - v_n^{\beta}] \\ &- \lambda\sum_{i=1}^{n} [(x_i + v_{i-1})^{\beta} - v_{i-1}^{\beta}] + (\beta - 1)\sum_{i=1}^{n} \ln(x_i + v_{i-1})] \end{aligned} \tag{15}$$

Where v_i can be obtained by Eqn. (11).

3.2.2 Multiple Systems

Suppose there are k systems:

$$\begin{aligned} L\{\text{data}|\lambda, \beta, q\} &= \prod_{l=1}^{k} f(t_{l,1}) f(t_{l,2} \mid t_{l,1}) \cdots f(t_{l,n_l} \mid t_{l,n_l-1})[R(T_l \mid t_{n_l})]^{\delta} \\ &= \prod_{l=1}^{k} \begin{bmatrix} \lambda^{n_l}\beta^{n_l} \left[e^{-\lambda[(T_l - t_{l,n_l} + v_{l,n_l})^{\beta} - v_{l,n_l}^{\beta}]} \right]^{\delta} \\ \prod_{i=1}^{n_l} [(x_{l,i} + v_{l,i-1})^{\beta-1} e^{-\lambda[(x_{l,i} + v_{l,i-1})^{\beta} - v_{l,i-1}^{\beta}]}] \end{bmatrix} \end{aligned}$$

Where:

$$\delta = \begin{cases} 0 & \text{If the test is failure truncated} \\ 1 & \text{If the test is time truncated} \end{cases}$$

Taking the natural log on both sides:

$$\begin{aligned} \log L = &\sum_{l=1}^{k} n_l(\ln\lambda + \ln\beta) - \lambda\delta\sum_{l=1}^{k}[T_l - t_{l,n_l} + v_{n_l})^{\beta} - v_{n_l}^{\beta}] \\ &- \lambda\sum_{l=1}^{k}\sum_{i=1}^{n_l}[(x_{l,i} + v_{l,i-1})^{\beta} - v_{l,i-1}^{\beta}] + (\beta - 1)\sum_{l=1}^{k}\sum_{i=1}^{n_l}\ln(x_{l,i} + v_{l,i-1}) \end{aligned} \tag{16}$$

There are three parameters (q, λ and β) that need to be estimated. However, there is no closed form mathematical solution. A numerical algorithm has been developed to solve both the single repairable system and the multiple repairable systems.

3.2.3 Grouped Data

The grouped data type is used for tests where the exact failure times are unknown and only the number of failures within a time interval is known (*e.g.* inspection data). As an example, this data type would be applicable when multiple units are run and the test units are inspected after predetermined time intervals and the number of failed units is recorded. The failed units are then subsequently repaired and put back into the test or removed.

The likelihood function is as follows:

$$L\{\text{data}|\lambda,\ \beta,\ \text{q}\} = \prod_{i=1}^{k} \frac{(\lambda(x_i + v_{i-1})^{\beta} - \lambda v_{i-1}{}^{\beta})^{n_i} e^{-(\lambda(x_i+v_{i-1})^{\beta} - \lambda v_{i-1}{}^{\beta})}}{n_i\ !}$$

Taking the natural log on both sides:

$$\text{Log } L = \sum_{i=1}^{k}\left[n_i \ln(\lambda(x_i + v_{i-1})^{\beta} - \lambda v_{i-1}{}^{\beta}) - (\lambda(x_i + v_{i-1})^{\beta} - \lambda v_{i-1}{}^{\beta}) - \ln n_i\,! \right] \quad (17)$$

In order to estimate the unknown parameters, we use a numerical method to maximize the log likelihood function like the single repairable system and the multiple repairable systems.

3.2.4 The Algorithm

Several algorithms have been used in the past for the maximization of the log-likelihood function such as the Newton search, genetic algorithms, annealing method, etc. The problem was approached with a form of the Newton search method, which is closely related to the Quasi-Newton method. This method was chosen because it is versatile, reliable, and provides quick convergence. The method maximizes the log-likelihood function (eq. (15), eq. (16) or eq. (7)) by taking Newton steps in order to bring its partial derivatives to zero. The full Newton step is always performed, since a quadratic convergence can be achieved once near the solution. At each iteration a check is performed to determine if the proposed step reduces the log-likelihood function. If not, a backtrack along the Newton direction is performed until an acceptable step is achieved. As in every optimization algorithm, the initial guesses for the parameters are very crucial. For this reason, much of the research focused in obtaining them. The initial guesses are obtained from the supplied data, thus increasing the probability of convergence to a global minimum (if it exists), and decreasing the number of iterations. For more detail see [9]

3.2.5 Confidence Bounds

Maximum likelihood estimation also permits the determination of *s*-confidence intervals for the unknown parameters and some metrics. The confidence bounds on the parameters and a number of other quantities, such as the reliability and the percentile, can be obtained based on the asymptotic theory for maximum likelihood estimates, most commonly referred to as the Fisher matrix bounds. The variance/covariance matrix of MLE of the parameters is obtained by taking the inverse of the Fisher matrix:

$$\Sigma = \begin{bmatrix} Var(\theta_1) & Cov(\theta_1,\theta_2) & \cdots & Cov(\theta_1,\theta_m) \\ Cov(\theta_1,\theta_2) & Var(\theta_2) & & \vdots \\ \vdots & & \ddots & \vdots \\ Cov(\theta_1,\theta_m) & \cdots & \cdots & Var(\theta_m) \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \Lambda}{\partial \theta_1^2} & \frac{\partial^2 \Lambda}{\partial \theta_1 \partial \theta_2} & \cdots & \frac{\partial^2 \Lambda}{\partial \theta_1 \partial \theta_m} \\ \frac{\partial^2 \Lambda}{\partial \theta_1 \partial \theta_2} & \frac{\partial^2 \Lambda}{\partial \theta_2^2} & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 \Lambda}{\partial \theta_1 \partial \theta_m} & \cdots & \cdots & \frac{\partial^2 \Lambda}{\partial \theta_m^2} \end{bmatrix}^{-1} \quad (18)$$

Using the delta method, the asymptotic variance of other metrics (for example, failure intensity) is given by $AV = \hat{D}\ '\sum \hat{D}$, Where $\hat{D}$ is the column vector of the partial derivatives of other metrics with respect to the parameters evaluated at the estimated parameters.

3.3 Example 1

To illustrate the general application of this model, consider a system tested for T=395.2 hours with the 56 failure times given in Table 1. The first failure was recorded at .7 hours into the test, the second failure was recorded 3 hours later at 3.7. The last failure occurred at 395.2 hours into the test and the system was removed from the test.

Table 1. Failure Data for a Repairable System [8]

0.	63.	125	244	315	366
3.	72.	133	249	317	373
1	99.	151	250	320	379
1	99.	163	260	324	389
1	100	164	263	324	394
2	102	174	273	342	395
4	112	177	274	350	
5	112	191	282	355	
5	120	192	285	364	
5	121	213	304	364	

This failure data are failure truncated. Based on this data set, different ML estimates of λ, β and repair degree q can be calculated corresponding to different model assumptions. The results of the ML estimates are shown in Table 2. For the RP column, we assume the repair activities are perfect repairs and the failure intensity is as good as new. Thus, we can obtain the λ, β estimates and LKV using Weibull++ 6. For the NHPP column, we assume that the repair actions restore the system operating state to be as bad as old. Therefore λ and β can be estimated by Eqns. (4) and (5). RGA 6 provides an excellent tool to analyze this type of repairable system. For the GRP (Kijima I) and GRP (Kijima II) columns, we use MLE to obtain λ, β and q estimates using this paper's models. The coming Weibull++ 7 software will include these models. From Table 2, we notice that the NHPP model and Kijima I 's results are the same for this particular data set.

Table 2. Analysis Results Comparison

	RP(Weibull)	NHPP(RG)	GRP (Kijima I)	GRP
$\hat{\lambda}$	0.1469	0.206	0.2061	0.89442
$\hat{\beta}$	0.9846	0.937	0.9372	0.24725
$\hat{q}$	0.0	1.0	1.0	0.93156
L	-165.41533	-165.30626	-165.30626	-165.247296

From Table 2, based on *LKV*, the GRP (Kijima II) is the best fit for this data set. Figure 2 shows the cumulative number of failures and their 90% two-sided confidence bounds based on the Kijima II model. Cumulative number of failures depends on the virtual time.

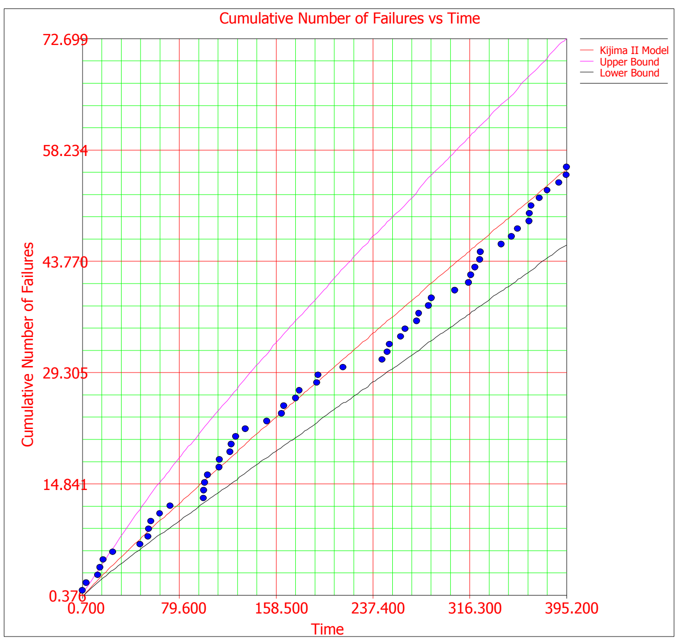

Figure 2. Cumulative Number of Failures and Two-Sided 90% Confidence Bounds

3.4 Example 2

Suppose K = 6 systems are observed during [0, T_i], $i = 1, \cdots, k$. That is, the data are time truncated with $T_1 = 8760$, $T_2 = 5000$, $T_3 = 6200$, $T_4 = 1300$, $T_5 = 2650$ and $T_5 = 500$. Failure data are given in Table 3.

Table 3. Failure Data for Repairable Systems

	System 1	System 2	System 3	System 4	System 5	System 6
Start	0	0	0	0	0	0
End	8760	5000	6200	1300	2650	500
1	2227.08	772.9542	900.9855	411.407	688.897	105.824
2	2733.229	1034.458	1289.95	1122.74	915.101	
3	3524.214	3011.114	2689.878			
4	5568.634	3121.458	3928.824			
5	5886.165	3624.158	4328.317			
6	5946.301	3758.296	4704.24			
7	6018.219		5052.586			
8	7202.724		5473.171			

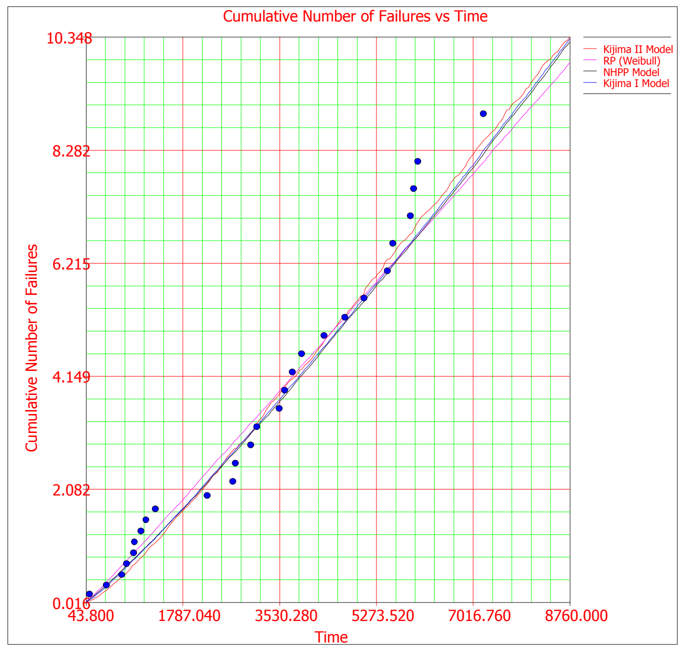

Figure 3. Cumulative Number of Failures vs Time

Table 4. Analysis Results Comparison

	RP(We	NHPP(RG)	GRP (Kijima I)	GRP
$\hat{\lambda}$	0.0004	0.000	0.00018	0.000068
$\hat{\beta}$	1.1409	1.113	1.23863	1.358201
$\hat{q}$	0.0	1.0	0.10599	0.552159
L	-210.43657	-210.57793	-210.30652	-209.95711

This failure data set is from multiple repairable systems. We can estimate λ, β, q and the cumulative number of failures utilizing the data from all six systems in Table 3. Table 4 shows the results of ML estimates based on different models .

From Table 4, based on *LKV*, the GRP (Kijima II) is the best fit for this data set. Figure 3 shows the cumulative number of failures based on all four models. We can see that the Kijima II model fits these multiple repairable systems very well and provides very promising results.

4 CONCLUSION

In this paper, we explored the *general renewal processes* based on the Weibull distribution for representing the reliability of complex repairable systems. The emphasis has been on solving problems with different types of data through model fitting and validation. An systematic MLE method is proposed for the parameters of the GRP, by assuming values of the repair effectiveness parameter of 0 and 1, the traditional ML estimators for NHPP and PRP can be obtained. Examples and procedures specifically illustrating these methods were given for two real world situations. In addition to maximum likelihood estimation methods, confidence interval procedures were discussed and illustrated by numerical examples. The proposed method provides excellent predictions with the potential of becoming very useful in practice and of leading to further generalization of repairable systems analysis.

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