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# D-Optimal Reliability Test Design for Two-Stress Accelerated Life Tests

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**Abstract** – Multiple-stress accelerated life tests (ALTs) are often employed for reliability testing of electronic parts. It is important to derive the optimal designs for these tests. However, most of the existing literature on experimental design of ALTs discusses only single-stress tests. In this paper we formulate the D-optimal design for two-stress ALTs with log-normal failure time distribution. It is shown that an efficient testing plan can be obtained with required estimation precision. ALT experimental designs for both completed failure time data and time censored data are discussed.

**Keywords** – Accelerated life testing; Design of experiments; D-optimality; Multiple stresses

## I. INTRODUCTION

Many electronic parts are expected to have very long life. Accelerated life testing (ALT) is a proactive way of obtaining product failure information within a reasonable time frame. In ALT products are tested under stresses, such as temperature, voltage, humidity, load cycles per unit time, etc., that are much more severe than in a normal-use condition. The testing can be used not only for predicting product's reliability but also studying the effects of various stress variables on product's lifetime [1-2]. Design and analysis of accelerated life tests involve stress and stress level selection, sample size and testing time determination, allocation of testing units, and handling censored failure time data, etc. It is typically more complicated than a traditional design of experiments (DOE).

Most of previous work on designing ALT plans focuses on ALTs with a single stress. Planning ALTs with two or more experimental factors was first studied in [3], where a linear model without interaction between factors was assumed. A common criterion of optimal test plan is to minimize the uncertainty of a quartile on failure time distribution under the product's normal-use condition. This leads to a degenerated type of design, in which multiple stresses are treated as a single stress and the designed test points are generated such that the effect of multiple stresses are the same as a single combined "stress". This strategy works well on planning ALTs for product's reliability demonstration, but may not be good at efficiently exploring the effect of each stress factor on product's lifetime.

Optimal designs of ALTs with multiple stresses were also discussed in [4-5]. In [4] it was assumed that the product's failure time follows an exponential distribution and the mean time to failure depends on the stresses according to the generalized Eyring law, in which the interaction between two stresses are modeled. A proportional hazard (PH) model based ALT plan was proposed in [5] and the optimality criterion was the variance of reliability estimation over the product's use period.

In this paper we discuss the D-optimal design of ALTs with two stress factors and their interaction. Nevertheless, the proposed method is a general method and can be easily extended to tests with more than two stresses. We assume that two stress levels are pre-specified for each stress before experiments and coded as -1 and +1. In two stress-level ALTs, the higher stress level is typically the highest possible stress that can be applied on the product without inducing unwanted failure mechanisms; while the lower stress is the one that can be determined by considering the precision of the estimation of reliability metric at normal-use stress. [2] provided some discussions on how to determine the optimal lower stress level and some practical considerations that may be used. The objective of D-optimality is to minimize the estimation uncertainty of model parameters, which is different from [3-4]. Therefore, with the D-optimal design the effect of each factor on product's lifetime can be better investigated. It is particularly useful for selecting significant factors.

## II. METHODOLOGY

The D-optimality in design of experiments is concerned with the precision of model parameter estimation. It maximizes the determinant of Fisher information matrix by selecting an appropriate design matrix. Typically with a linear model the optimization can be carried out by using least square method. However, in reliability testing censored data are often encountered, which causes difficulties in data analysis. In this section, we derive likelihood functions to obtain the Fisher information matrix of a test plan. The pattern of an optimal design can be observed by examining this matrix.

We assume that failure time follows a log-normal distribution and stresses affect the location parameter of the distribution only. Two-level factorial design is used here. We refer each factor-level combination as a "run".

Denote  $X_{i,1}$  and  $X_{i,2}$  the stress levels for stress factor 1 and stress factor 2 at the  $i$ -th run, respectively. Let  $t_{i,j}$  be the failure time observed at the  $i$ -th run for the  $j$ -th specimen. For a single failure observation, the log-failure time can be modeled by

$$Y_{i,j} = \ln t_{i,j} = \beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \beta_{12} X_{i,1} X_{i,2} + \sigma \varepsilon_{ij} \quad (1)$$

where  $Y_{i,j}$  is the log transformed failure time response,  $\beta$  is the parameter vector which quantifies the effects of stresses on specimen's failure time,  $\sigma$  is the variance parameter and  $\varepsilon_{i,j}$  is the random error,  $\varepsilon_{i,j} \sim N(0,1)$ .

Define a standardized log failure time as

$$z_{i,j} = \frac{Y_{i,j} - (\beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \beta_{12} X_{i,1} X_{i,2})}{\sigma}$$

Thus,  $z_{i,j} \sim N(0,1)$ .

The probability density function of a single log failure time observation  $Y_{i,j}$  is

$$f(y_{i,j}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2} [y_{i,j} - (\beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \beta_{12} X_{i,1} X_{i,2})]^2} = \frac{1}{\sqrt{2\pi}\sigma} e^{-z_{ij}^2/2}$$

We can get the log-likelihood functions for a single log failure time observation, which is

$$L_{i,j}(\beta, \sigma) = -\ln \sigma - \frac{1}{2} \ln 2\pi - \frac{z_{i,j}^2}{2} \quad (2)$$

Taking the first-order derivation with respect to each parameter, we have

$$\frac{\partial L_{ij}}{\partial \beta_0} = \frac{z_{ij}}{\sigma}, \quad \frac{\partial L_{ij}}{\partial \beta_1} = \frac{z_{ij} x_{i1}}{\sigma}, \quad \frac{\partial L_{ij}}{\partial \beta_2} = \frac{z_{ij} x_{i2}}{\sigma},$$

$$\frac{\partial L_{ij}}{\partial \beta_{12}} = \frac{z_{ij} x_{i1} x_{i2}}{\sigma}, \quad \frac{\partial L_{ij}}{\partial \sigma} = \frac{z_{ij}^2 - 1}{\sigma}$$

It is easy to obtain the second-order derivatives. Given that  $E[z_{i,j}] = 0$  and  $E[z_{i,j}^2] = 1$ , the expected Fisher information matrix for a single failure time observation is

$$F_{ij} = \frac{1}{\sigma^2} \begin{bmatrix} 1 & x_{i1} & x_{i2} & x_{i1}x_{i2} & 0 \\ & x_{i1}^2 & x_{i1}x_{i2} & x_{i1}^2x_{i2} & 0 \\ & & x_{i2}^2 & x_{i1}x_{i2}^2 & 0 \\ & & & x_{i1}^2x_{i2}^2 & 0 \\ \text{syn} & & & & 2 \end{bmatrix} \quad (3)$$

Let  $n_i$  be the number of specimens at the  $i$ -th run and  $n = \sum_i n_i$ , is the total number of specimens to be tested.

Given that the two stress levels of each stress are pre-specified and coded as -1 and +1, we have  $x_{i1}^2 = x_{i2}^2 = 1$ . When the complete failure data from

ALTs are available, the overall expected Fisher information matrix becomes

$$F = \sum_i \sum_j F_{ij} = \frac{1}{\sigma^2} \begin{bmatrix} n & \sum_i n_i x_{i1} & \sum_i n_i x_{i2} & \sum_i n_i x_{i1} x_{i2} & 0 \\ & n & \sum_i n_i x_{i1} x_{i2} & \sum_i n_i x_{i2} & 0 \\ & & n & \sum_i n_i x_{i1} & 0 \\ & & & n & 0 \\ \text{syn} & & & & 2n \end{bmatrix} \quad (4)$$

From (4) one can see that when complete failure time data are available, i.e., there is not restriction on the ALT testing time, the expected Fisher information matrix is a function of  $n_i$ , the allocation of specimens at each run, and  $X$ , the design matrix of experiments, so to maximize the determinant of  $F$  means to find the optimal design with optimal specimen allocation scheme. As the typical D-optimal design of linear model, the optimal experiment points are located at the boundary of feasible region.

Now we consider the time censored failure data. Let  $\tau$  be the maximum allowable log testing time, i.e., for any specimen which does not fail by the time  $e^\tau$ , its failure time is censored. Define  $s_i$  as

$$s_i = \frac{\tau - (\beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \beta_{12} X_{i,1} X_{i,2})}{\sigma}$$

So the probability of a specimen survives at time  $\tau$  is

$$R(\tau) = 1 - \int_{-\infty}^{\tau} f(y_{ij}) dy_{ij} = 1 - \int_{-\infty}^{s_i} f(z_{ij}) dz_{ij} = 1 - \Phi(s_i)$$

where  $\Phi(s_i)$  (we simplify the notation as  $\Phi_i$  in the following text) is the cumulative standard normal distribution function.

We use an indicator variable

$$I_{ij} = \begin{cases} 1 & \text{if } Y_{ij} \leq \tau \\ 0 & \text{if } Y_{ij} > \tau \end{cases}$$

So the likelihood function from a single observation (either a failure or a censor time) is

$$l_{ij} = f(y_{ij})^{I_{ij}} R(\tau)^{1-I_{ij}}$$

and the log-likelihood function becomes

$$L_{i,j}(\beta, \sigma) = I_{ij} \left( -\ln \sigma - \frac{1}{2} \ln 2\pi - \frac{z_{i,j}^2}{2} \right) + (1 - I_{ij}) \ln(1 - \Phi_i) \quad (5)$$

Again, the first-order derivatives can be obtained as

$$\frac{\partial L_{ij}}{\partial \beta_0} = \frac{1}{\sigma} \left[ I_{ij} z_{ij} + (1 - I_{ij}) \frac{\phi_i}{\Phi_i} \right];$$

$$\frac{\partial L_{ij}}{\partial \beta_1} = \frac{1}{\sigma} \left[ I_{ij} z_{ij} x_{i1} + (1 - I_{ij}) \frac{\phi_i x_{i1}}{1 - \Phi_i} \right];$$

$$\frac{\partial L_{ij}}{\partial \beta_2} = \frac{1}{\sigma} \left[ I_{ij} z_{ij} x_{i2} + (1 - I_{ij}) \frac{\phi_i x_{i2}}{1 - \Phi_i} \right];$$

$$\frac{\partial L_{ij}}{\partial \beta_{12}} = \frac{1}{\sigma} \left[ I_{ij} z_{ij} x_{i1} x_{i2} + (1 - I_{ij}) \frac{\phi_i x_{i1} x_{i2}}{1 - \Phi_i} \right];$$

$$\frac{\partial L_{ij}}{\partial \sigma} = \frac{1}{\sigma} \left[ I_{ij} (z_{ij}^2 - 1) + (1 - I_{ij}) \frac{\phi_i s_i}{1 - \Phi_i} \right]$$

where  $\phi_i = \phi(s_i)$ , is the pdf of standard normal distribution at the censor time.

To derive the expected Fisher information matrix, the second-order derivatives of the log-likelihood function must be calculated. Given  $E[z_{ij}] = 0$ ,  $E[I_{ij}] = \Phi_i$ , and assume  $z_{ij}$  and  $I_{ij}$  are independent, we have

$$\begin{aligned} E \left[ \frac{\partial L_{ij}^2}{\partial^2 \beta_0} \right] &= E \left[ \frac{\partial L_{ij}^2}{\partial^2 \beta_1} \right] = E \left[ \frac{\partial L_{ij}^2}{\partial^2 \beta_2} \right] = E \left[ \frac{\partial L_{ij}^2}{\partial^2 \beta_{12}} \right] \\ &= -\frac{1}{\sigma^2} \left[ \Phi_i - \phi_i \left( s_i - \frac{\phi_i}{1 - \Phi_i} \right) \right]; \\ E \left[ \frac{\partial L_{ij}^2}{\partial \beta_0 \partial \beta_1} \right] &= -\frac{x_{i1}}{\sigma^2} \left[ \Phi_i - \phi_i \left( s_i - \frac{\phi_i}{1 - \Phi_i} \right) \right]; \\ E \left[ \frac{\partial L_{ij}^2}{\partial \beta_0 \partial \beta_2} \right] &= -\frac{x_{i2}}{\sigma^2} \left[ \Phi_i - \phi_i \left( s_i - \frac{\phi_i}{1 - \Phi_i} \right) \right]; \\ E \left[ \frac{\partial L_{ij}^2}{\partial \beta_0 \partial \beta_{12}} \right] &= -\frac{x_{i1} x_{i2}}{\sigma^2} \left[ \Phi_i - \phi_i \left( s_i - \frac{\phi_i}{1 - \Phi_i} \right) \right]; \\ E \left[ \frac{\partial L_{ij}^2}{\partial \beta_1 \partial \beta_2} \right] &= -\frac{x_{i1} x_{i2}}{\sigma^2} \left[ \Phi_i - \phi_i \left( s_i - \frac{\phi_i}{1 - \Phi_i} \right) \right]; \\ E \left[ \frac{\partial L_{ij}^2}{\partial \beta_1 \partial \beta_{12}} \right] &= -\frac{x_{i2}}{\sigma^2} \left[ \Phi_i - \phi_i \left( s_i - \frac{\phi_i}{1 - \Phi_i} \right) \right]; \\ E \left[ \frac{\partial L_{ij}^2}{\partial \beta_2 \partial \beta_{12}} \right] &= -\frac{x_{i1}}{\sigma^2} \left[ \Phi_i - \phi_i \left( s_i - \frac{\phi_i}{1 - \Phi_i} \right) \right]; \\ E \left[ \frac{\partial L_{ij}^2}{\partial^2 \sigma} \right] &= -\frac{1}{\sigma^2} \left[ 2\Phi_i - \phi_i s_i \left( s_i^2 - 1 - \frac{\phi_i s_i}{1 - \Phi_i} \right) \right]; \\ E \left[ \frac{\partial L_{ij}^2}{\partial \beta_0 \partial \sigma} \right] &= \frac{1}{\sigma^2} \left[ \phi_i s_i \left( s_i - \frac{\phi_i}{1 - \Phi_i} \right) \right]; \end{aligned}$$

$$E \left[ \frac{\partial L_{ij}^2}{\partial \beta_1 \partial \sigma} \right] = \frac{x_{i1}}{\sigma^2} \left[ \phi_i s_i \left( s_i - \frac{\phi_i}{1 - \Phi_i} \right) \right];$$

$$E \left[ \frac{\partial L_{ij}^2}{\partial \beta_2 \partial \sigma} \right] = \frac{x_{i2}}{\sigma^2} \left[ \phi_i s_i \left( s_i - \frac{\phi_i}{1 - \Phi_i} \right) \right];$$

$$E \left[ \frac{\partial L_{ij}^2}{\partial \beta_{12} \partial \sigma} \right] = \frac{x_{i1} x_{i2}}{\sigma^2} \left[ \phi_i s_i \left( s_i - \frac{\phi_i}{1 - \Phi_i} \right) \right]$$

Therefore, for the ALT tests of  $n_i$  specimens to be tested at the  $i$ -th run and with time censored data, the expected Fisher information matrix is given by (6), where

$$A_i = \Phi_i - \phi_i \left( s_i - \frac{\phi_i}{1 - \Phi_i} \right).$$

Unlike (4), (6) shows that with time censored data the expected Fisher information matrix depends on the value of parameters  $\beta$  and  $\sigma$ . Clearly some prior knowledge of  $\beta$  and  $\sigma$  should be known in order to design an optimal test plan. This knowledge may be obtained from engineering expertise and/or from preliminary data analysis, and these values are referred as the planned parameter values.

### III. EXAMPLES

Both temperature and humidity affect the life time of an electronic part. A 2X2 full factorial design is applied on the ALTs to test system reliability of this product. The design matrix is given as

$i$	$x_1$	$x_2$	$x_1 x_2$
1	-1	-1	+1
2	-1	+1	-1
3	+1	-1	-1
4	+1	+1	+1

So,

$$\begin{aligned} \sum_i n_i x_{i1} &= n_1(-1) + n_2(-1) + n_3 + (n - n_1 - n_2 - n_3) \\ &= n - 2(n_1 + n_2); \end{aligned}$$

$$F = \frac{1}{\sigma^2} \begin{bmatrix} \sum_i n_i A_i & \sum_i n_i x_{i1} A_i & \sum_i n_i x_{i2} A_i & \sum_i n_i x_{i1} x_{i2} A_i & \sum_i n_i (A_i - \Phi_i) s_i \\ & \sum_i n_i A_i & \sum_i n_i x_{i1} x_{i2} A_i & \sum_i n_i x_{i2} A_i & \sum_i n_i (A_i - \Phi_i) s_i x_{i1} \\ & & \sum_i n_i A_i & \sum_i n_i x_{i1} A_i & \sum_i n_i (A_i - \Phi_i) s_i x_{i2} \\ & & & \sum_i n_i A_i & \sum_i n_i (A_i - \Phi_i) s_i x_{i1} x_{i2} \\ & \text{syn} & & & \sum_i n_i \left[ 2\Phi_i - \phi_i s_i \left( s_i^2 - 1 - \frac{\phi_i s_i}{1 - \Phi_i} \right) \right]_i \end{bmatrix} \quad (6)$$

$$\sum_i n_i x_{i2} = n - 2(n_1 + n_3);$$

$$\sum_i n_i x_{i1} x_{i2} = n - 2(n_2 + n_3)$$

From (4), the F matrix of  $\beta$  becomes

$$F_\beta = \frac{1}{\sigma^2} \begin{bmatrix} n & n-2(n_1+n_2) & n-2(n_1+n_3) & n-2(n_2+n_3) \\ & n & n-2(n_2+n_3) & n-2(n_1+n_3) \\ & & n & n-2(n_1+n_2) \\ & & & n \\ \text{syn} & & & \end{bmatrix}$$

In order to maximize the information we can obtain from the experiment and thus to reduce the parameter estimation uncertainty, we need to find the optimal value of  $n_i$  to maximize the determinant of the  $F$  matrix. Because this matrix is a symmetric matrix and the notation of  $n_i$  is exchangeable, without other constraints the optimal specimen allocation will be  $n_1 = n_2 = n_3 = n_4 = (1/4)n$ . However, if we have budget constraints and the testing costs at different runs are different, we need to use numerical methods to find the optimal solution.

For instance, let a total of 40 specimens be available and the cost of each run is listed in Table 1. Because of the limit of the testing equipment, there are constraints on the total number of specimens that can be placed on each run. It is desired that the total cost of ALTs is less than 150.

Table 1. The ALT design matrix and constraints

Temperature	Humidity	Interaction	Available Test Units	Cost/unit
-1	-1	+1	$n_1 \leq 15$	5
-1	+1	-1	$n_2 \leq 10$	4
+1	-1	+1	$n_3 \leq 20$	3
+1	+1	+1	$n_4 \leq 15$	10

The optimization is set up as

$$\text{Max } |F_\beta|$$

S.T.

$$n_1 \leq 15; \quad n_2 \leq 10; \quad n_3 \leq 20; \quad n_4 \leq 15;$$

$$n_1 + n_2 + n_3 + n_4 \leq 40;$$

$$5n_1 + 4n_2 + 3n_3 + 10n_4 \leq 150$$

Assume that the testing time is long enough for obtaining the complete failure time data. Similar to the regular D-optimality in DOE, the optimal test plan will locate each test point at the boundary of the feasible region. The solution for the above problem is given in Table 2.

Table 2. The solution for D-optimal design with constraints

	Test Unit	cost
$n_1$	7	35
$n_2$	9	36
$n_3$	13	39
$n_4$	4	40
Total	33	150

The corresponding Fisher information matrix of  $\beta$  is

$$\begin{bmatrix} 33 & 1 & -7 & -11 \\ 1 & 33 & -11 & -7 \\ -7 & -11 & 33 & 1 \\ -11 & -7 & 1 & 33 \end{bmatrix}$$

#### IV. CONCLUSION

In this paper the D-optimal designs for two-stress two-level ALT were studied for both complete failure time data and time censored data. It was assumed that the failure time distribution follows a log-normal distribution and the stress factor only affect the location parameter of the log-normal distribution. Similar to a typical D-optimal DOE, if no constraints are considered, the optimal ALT tends to locate experimental points evenly at the corner of the feasible region of stress factors. It is because the objective of experiments is to minimize the uncertainty in the estimation of model parameters, which are the effects of stress factors on product's mean life. If the constraints of test units, testing equipment and budget have to be considered, the D-optimal ALT plan can be found by numerical methods. When the testing time is limited, the D-optimal ALT plan will depend on the prior information of the parameters to be estimated. The proposed method is different from previous studies in that it emphasizes on the investigation and understanding of stress effects on product failure, instead of merely demonstrating product reliability under its normal-use condition. It will be more useful for reliability engineers at product design stage.

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